國立政治大學 101 學年度轉學生招生考試試題

考試科目	高等微積分	系別	統計學系	考試時間	7 月 6 日(五) 第 一 節

1. (40pts). For each of the following statements, determine whether it is true or false. Do not give explanation.

   (a) Given a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers, then $\lim_{n\to\infty} a_n = a$ if and only if any neighborhood of a contains infinitely many terms of the sequence $\{a_n\}$.

   (b) The set of rational numbers $\mathcal{Q}$ is dense in the real line $\mathcal{R}$.

   (c) Suppose f and g are both uniformly continuous on $\mathcal{R}$. If $f(x) < g(x)$ for all $x < 0$, then $f(0) = \lim_{x\to 0^-} f(x) < \lim_{x\to 0^-} g(x) = g(0)$.

   (d) If $f : \mathcal{R} \longrightarrow \mathcal{R}$ is differentiable and strictly increasing, then $f'(x) > 0$ for all $x \in \mathcal{R}$.

   (e) Consider two continuous functions $f, g : \mathcal{R} \longrightarrow \mathcal{R}$. Then the set $\{x \in \mathcal{R} : |f(x) - g(x)| > 1\}$ is open in $\mathcal{R}$.

   (f) If two power series $\Sigma_{k=0}^{\infty} a_k x^k$ and $\Sigma_{k=0}^{\infty} b_k x^k$ have as their radius of convergence r_a and r_b respectively, then the power series $\Sigma_{k=0}^{\infty}(a_k + b_k)x^k$ has $\min(r_a, r_b)$ as its radius of convergence.

   (g) If $f : \mathcal{R}^2 \longrightarrow \mathcal{R}$ has first-order partial derivates and $\frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial y}(x,y) = 0$ for all $(x,y) \in \mathcal{R}^2$, then f is a constant function.

   (h) Consider the sequence of differentiable functions $\{f_n\}_{n=1}^{\infty}$ with $f_n : [0,1] \longrightarrow \mathcal{R}$. If f_n converges uniformly to a differentiable function $f : [0,1] \longrightarrow \mathcal{R}$ on $[0,1]$, then $\lim_{n\to\infty} f_n'(x) = f'(x)$ for all $x \in [0,1]$.

   (i) If $f : [0,1] \longrightarrow \mathcal{R}$ is monotone, then f is integrable on $[0,1]$.

   (j) Suppose $f : R = [0,1]\times[0,1] \longrightarrow \mathcal{R}$ is bounded. Then we have $\underline{\int\int}_R f \le \overline{\int_0^1} \left(\underline{\int_0^1} f(x,y)\mathrm{d}x\right) \mathrm{d}y \le \overline{\int\int}_R f$.

備註	試題隨卷繳交

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2. Please state the following definition.

   (a) (4pts). We say a series $\Sigma_{k=1}^{\infty} a_k$ is conditionally convergent provided that __________________.

   (b) (6pts). Let $f : [a,b] \to \mathcal{R}$ and $f_n : [a,b] \to \mathcal{R}$ for $n \in \mathcal{N}$. We say $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f on $[a,b]$ provided that __________________.

   (c) (6pts). Suppose $f : \mathcal{R}^2 \longrightarrow \mathcal{R}$ and $\mathbf{u} = (u_1, u_2)$ is a unit vector ($u_1^2 + u_2^2 = 1$). We say f has a directional derivative at the origin $(0,0)$ in the direction of $\mathbf{u}$ provided that __________________.

3. (12pts). Prove or disprove that if $f : (-1,1) \longrightarrow \mathcal{R}$ is continuous and bounded, then f is uniformly continuous on $(-1,1)$. Justify your answer.

4. (12pts). Prove or disprove that if $f : [-1,1] \longrightarrow \mathcal{R}$ is continuous, nonnegative and $\int_{-1}^{1} f = 0$, then $f(x) = 0$ for all $x \in [-1,1]$. Justify your answer.

5. (10pts). Suppose $f : \mathcal{R}^2 \longrightarrow \mathcal{R}$ and we already know that $f(x,y) = \frac{xy^2}{x^2+y^4}$ for $(x,y) \neq (0,0)$. Is it possible to define a suitable value for $f(0,0)$ such that f is continuous at $(0,0)$? Justify your answer.

6. (10pts). Given $f : [a,b] \longrightarrow \mathcal{R}$ and $g : [c,d] \longrightarrow \mathcal{R}$, define $h : [a,b] \times [c,d] \longrightarrow \mathcal{R}$ by $h(x,y) = f(x)g(y)$. Let $P_x = \{x_0 = a < x_1 < \ldots < x_n = b\}$ be a partition of $[a,b]$, $P_y = \{y_0 = c < y_1 < \ldots < y_m = d\}$ be a partition of $[c,d]$, and $P = P_x \times P_y$, a partition of $[a,b] \times [c,d]$. Please prove that $U(P,h) \leq U(P_x, f)U(P_y, g)$. (Note $U(P,h)$ is the upper (Darboux) sum of function h based on partition P.)

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考試科目	數理統計學	系別	統計系	考試時間	7月6日(五) 第二節

1. Calculate the integral $I = \int_{-\infty}^{\infty} \exp\left(\frac{-y^2}{2}\right) dy$. (10%)

2. Let X have the uniform distribution with p.d.f. $f(x) = 1, 0 < x < 1$, zero elsewhere. Find the distribution function of $Y = -2\ln X$. What is the p.d.f. of Y? (10%)

3. Let $X_1, X_2, \cdots$ be independent Bernoulli random variables, $X_i \sim Bin(1, p_i)$, and let $Y_n = \sum_{i=1}^{n} \frac{X_i - p_i}{n}$. Show that the sequence $Y_1, Y_2, \cdots$ converges stochastically to $c = 0$ as $n \to \infty$. (15%)

4. Let $\bar{X}$ and S^2 be the mean and the variance of a random sample of size 36 from a distribution that is $N(4, 144)$. Calculate $Pr(0 < \bar{X} < 6,\ 45 < S^2 < 140)$. (10%) (do not need to give the exact value.)

5. Assume that $X_i \sim Lognomal(\mu_i, \sigma_i^2)$, $i = 1,2,\cdots,n$ are independent. Find the distribution of

   (a) $\frac{Y_3}{Y_5}$. (b) $\prod_{i=1}^{n} Y_i^4$. (c) Find $E[\prod_{i=1}^{n} Y_i]$. (18%)

6. Let $X_1, X_2, \cdots, X_n$ be a random sample from $N(\mu, \sigma^2)$. Show that the sample mean $\bar{X}$ and each $X_i - \bar{X}$, $i = 1, 2, \cdots n$, are independent. (15%)

7. Let $X_1, X_2, \cdots, X_n$ be a random sample from a distribution with p.m.f $f(x) = \frac{1}{6}, x = 1, 2, \cdots, 6$, zero elsewhere. Let $Y = \min(X_i)$ and $Z = \max(X_i)$. Say that the joint distribution function of Y and Z is $G(y,z) = Pr(Y \le y, Z \le z)$, where y and z are nonnegative integers such that $1 \le y \le z \le 6$.

   (a) Show that $G(y,z) = F^n(z) - [F(z) - F(y)]^n$, $1 \le y \le z \le 6$, where F(x) is the distribution function associated with f(x).

   (b) Find the joint p.m.f. of Y and Z. (22%)

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