國立政治大學 100 學年度轉學生招生考試試題

考試科目	高等微積分	系別	統計學系	考試時間	7月8日(五)第1節

1. (40pts). For each of the following statements, determine whether it is true or false. Do not give explanation.

   (a) The set of rational numbers $\mathcal{Q}$ and the set of integers $\mathcal{Z}$ have the same cardinality.

   (b) If A_1, A_2, A_3,... are compact sets of $\mathcal{R}$, then $A \equiv \cup_{n=1}^{\infty} A_i$ is compact in $\mathcal{R}$.

   (c) Let $f : \mathcal{R} \longrightarrow \mathcal{R}$. if f is uniformly continuous on every interval $[m, m+1]$ for $m \in \mathcal{Z}$, then f is uniformly continuous on $\mathcal{R} = \cup_{m=-\infty}^{\infty}[m, m+1]$.

   (d) Let $f, g : [a, b] \longrightarrow \mathcal{R}$. If f and g are both Riemann integrable on $[a, b]$, then so is $h : [a, b] \longrightarrow \mathcal{R}$ defined by $h(x) = \max(f(x), g(x))$.

   (e) Suppose $f : \mathcal{R} \longrightarrow \mathcal{R}$ is differentiable. Then $h : \mathcal{R} \longrightarrow \mathcal{R}$ defined by $h(x) = \begin{cases} \frac{f(x)-f(0)}{x} & \text{if } x \neq 0 \\ f'(0) & \text{if } x = 0 \end{cases}$ is a continuous function.

   (f) If $\{f_n\}_{n=0}^{\infty}$ is a sequence of continuous functions on $[0, 1]$ converging pointwise to a continuous function f on $[0, 1]$, then the convergence is uniform.

   (g) If the power series $\Sigma_{k=0}^{\infty} a_k (x+2)^k$ diverges when $x = -\frac{5}{2}$, then it also diverges when $x = -\frac{1}{2}$.

   (h) If $\Sigma_{k=0}^{\infty} a_k$ converges to S absolutely, then any rearrangement of $\Sigma_{k=0}^{\infty} a_k$ converges to S absolutely as well.

   (i) Suppose $\{f_n\}_{n=0}^{\infty}$ is a sequence of continuously differentiable functions on (a, b) that converges uniformly to a differentiable function f on (a, b), then $\{f_n'\}_{n=0}^{\infty}$ converges pointwise to f' on (a, b).

   (j) Let $f : [a, b] \times [c, d] \longrightarrow \mathcal{R}$ be bounded. Then $\overline{\int}_{[a,b]\times[c,d]} f dxdy \leq \overline{\int}_a^b (\overline{\int}_c^d f dy) dx$.

備註	試題隨卷繳交

【請注意，背面還有試題。】

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2. Suppose $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are both Cauchy sequences.

   (a) (3pts). Please write down the definition of a Cauchy sequence for $\{a_n\}_{n=1}^{\infty}$.

   (b) (5pts). Please Prove that $\{a_n\}$ is bounded.

   (c) (7pts). Please prove that $\{a_nb_n\}_{n=1}^{\infty}$ is a Cauchy Sequence.

3. (6pts). Cauchy's mean value theorem is stated as follows: if f and g are continuous on $[a,b]$ and differentiable on (a,b), then there exists $c \in (a,b)$ such that $f'(c)(g(b)-g(a)) = g'(c)(f(b)-f(a))$. What is wrong with the following proof for the Cauchy's mean value theorem? (Only one place and do not need to correct it.)

   Since f and g satisfy the hypotheses of the mean value theorem, there exists $c \in (a,b)$ such that $f(b)-f(a) = f'(c)(b-a)$ and $g(b)-g(a) = g'(c)(b-a)$. Then $f'(c)(g(b)-g(a)) = \frac{f(b)-f(a)}{b-a}(g'(c)(b-a)) = g'(c)(f(b)-f(a))$.

4. (10pts). Suppose $f:[a,b] \longrightarrow \mathcal{R}$ is continuous. Please prove that $f \geq 0$ if $h:[a,b] \longrightarrow \mathcal{R}$ defined by $h(x) = \int_a^x f(t)dt$ is increasing on $[a,b]$.

5. (6pts). Suppose $\Sigma_{k=0}^{\infty} a_kx^k$ and $\Sigma_{k=0}^{\infty} b_kx^k$ have as their radius of convergence both R. Then will the power series $\Sigma_{k=0}^{\infty}(a_k+b_k)x^k$ have radius of convergence R? Justify your answer.

6. Let $f:\mathcal{R}^2 \longrightarrow \mathcal{R}$ defined by $f(x,y) = \begin{cases} \frac{x^3y}{x^6+y^3} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) \neq (0,0) \end{cases}$

   (a) (6pts). Please find the partial derivatives $f_x(0,0)$ and $f_y(0,0)$. Show your work.

   (b) (7pts). Please find the directional derivative $f_u(0,0)$, where $u = (u_x, u_y)$ and $u_x^2 + u_y^2 = 1$. Show your work.

   (c) (10pts). Please show that f is not continuous at $(0,0)$.

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考試科目	數理統計學	系別	統計	考試時間	7月8日(五)第二節

1. Let X have the hypergeometric distribution with parameters n, M and N, then for each $x=0,1,\cdots,n$, and as $N\to\infty$ and $M\to\infty$ with $M/N\to p$, a positive constant

(10) $$\lim_{N\to\infty}\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{x}}=\binom{n}{x}p^x(1-p)^{n-x}$$

2. Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from $U(0,1)$

(a) Identify the distribution of the k^{th} order statistic $X_{k:n}$

(20) (b) Find the pdf of Range $R_n = X_{n:n} - X_{1:n}$

3. Let $X_1, X_2, \cdots, X_n, \cdots$ be a random sample of size n from a distribution with finite mean μ, and variance σ^2. Let $\overline{X}_n$ and S_n^2 be the sample mean and sample variance.

(a) Show that $S_n^2 \xrightarrow{P} \sigma^2$

(20) (b) Show that $\dfrac{\sqrt{n}(\overline{X}_n-\mu)}{S_n} \xrightarrow{d} Z \sim N(0,1)$

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4. Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from $U(0, \theta)$

(5) (a) Find MLE of θ.

(20) (b) Find UMVUE of θ. by finding a complete and sufficient statistic T_n first, then find a ft of T_n and show that it is unbiased.

(20) (c) Find UMP test for testing $H_0: \theta = \theta_0$ vs. $H_a: \theta > \theta_0$ at significance level α.

(5) (d) Find $100(1-\alpha)\%$ Confidence Interval of θ.

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