國立政治大學 109 學年度 碩士暨碩士在職專班 招生考試試題

考試科目	基礎數學	系所別	統計學系	考試時間	2 月 7 日(五) 第一 節

1. (30%) Suppose the correlation matrix of a random vector (X_1, X_2, X_3) is given as

$$\Sigma = \begin{pmatrix} 1.0 & 0.5 & -0.5 \\ 0.5 & 1.0 & 0 \\ -0.5 & 0 & 1.0 \end{pmatrix}.$$

   (a) (6%) Find the eigenvalues of Σ.

   (b) (9%) Find bases for the eigenspaces associated with each eigenvalues of Σ.

   (c) (6%) Find the diagonal matrix D and the orthogonal matrix P to orthogonally diagonalize Σ.

   (d) (3%) Find the spectral decomposition of Σ.

   (e) (6%) Show that Σ is positive semi-definite and find $\Sigma^{1/2}$.

2. (20%) Suppose A is a $n \times n$ symmetric matrix.

   (a) (10%) Show that vectors corresponding to distinct eigenspaces of A are orthogonal.

   (b) (10%) Consider the following definition. Show that

$$< \mathbf{u}, \mathbf{v} >= \mathbf{u}^T A \mathbf{v}$$

   defines an inner product in R^n.

   Definition 1 *An* **inner product** *on a vector space* $\mathbf{V}$ *is an operation that assigns to every pair of vectors* $\mathbf{u}, \mathbf{v} \in \mathbf{V}$ *a real number* $< \mathbf{u}, \mathbf{v} >$ *such that the following properties hold for all* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}$ *and* $c \in R$.

   i. $< \mathbf{u}, \mathbf{v} >=< \mathbf{v}, \mathbf{u} >$.

   ii. $< \mathbf{u}, \mathbf{v} + \mathbf{w} >=< \mathbf{u}, \mathbf{v} > + < \mathbf{u}, \mathbf{w} >$.

   iii. $< c\mathbf{u}, \mathbf{v} >= c < \mathbf{u}, \mathbf{v} >$.

   iv. $< \mathbf{u}, \mathbf{u} > \geq 0$ *and* $< \mathbf{u}, \mathbf{u} >= 0$ *if and only if* $\mathbf{u} = \mathbf{0}$.

   A vector space with an inner product is called an **an inner product space**.

備註	一、作答於試題上者，不予計分。 二、試題請隨卷繳交。

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3. (5 %; 1% for each part) Write down the final answers only.

   (a) Find $\frac{d}{dx}(x^2+3x+1)$.

   (b) Find $\frac{d}{dx}e^x$.

   (c) Find $\frac{d}{dx}(\ln(x)+x\sin(x))$.

   (d) Find $\frac{d}{dx}\frac{1}{2+\cos(x)}$.

   (e) Find $\frac{d}{dx}\sin(\cos(x))$.

- Note. For Problems 4 – 7, you need to show your work in the solutions. Writing down the final answers only for these problems is not enough to receive any points.

4. (20%) Let $I_n(t)=\int_0^t x^n\cos(x)dx$ and $J_n(t)=\int_0^t x^n\sin(x)dx$ for $n\geq 0$ and $t\in(-\infty,\infty)$.

   (a) (14%) Express $I_{n+1}(t)$ and $J_{n+1}(t)$ in terms of $I_n(t)$, $J_n(t)$, n and t.

   (b) (6 %) Compute $I_0(\pi)$ and $J_0(\pi)$, and then find $I_1(\pi)$ and $J_1(\pi)$ using the expressions in Part (a).

5. (10 %) Let $D(r)=\{(x,y): x^2+y^2\leq r^2\}$ for $r>0$.

   (a) (7%) Find $\int_{D(r)} e^{-x^2-y^2}d(x,y)$ for $r>0$.

   (b) (3 %) Find $\lim_{r\to\infty}\int_{D(r)} e^{-x^2-y^2}d(x,y)$ based on your answer for Part (a).

6. (10 %) Let $f(x,y)=x^2+\sin(xy)+y^2$. Does f have a local minimum or a local maximum at $(x,y)=(0,0)$? Justify your answer.

7. (5%) Let $\phi(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ for $x\in(-\infty,\infty)$ and $G(x)=\int_x^\infty \phi(t)dt$ for $x\in(-\infty,\infty)$. Find $\lim_{x\to\infty}\frac{xG(x)}{\phi(x)}$.

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國立政治大學 109 學年度 碩士暨碩士在職專班 招生考試試題

考 試 科 目	數理統計學	系 所 別	統計學系	考 試 時 間	2 月 7 日(五) 第二 節

1. (a) Let $X_1, X_2, \cdots, X_n$ be i.i.d. uniform random variables in the interval $(0,1)$. Find the density function of the range of $(X_1, X_2, \cdots, X_n)$. (10%)

   (b) Let $X_1, X_2, \cdots, X_n$ be i.i.d. random variables with an absolutely continuous monotone increasing distribution function F and also let $(Y_1, Y_2, \cdots, Y_n)$ denote the corresponding ordered statistic. Prove that $\mathrm{F}(X_1)$ is distributed uniformly in the interval $(0,1)$ and hence show how you can use the result you have obtained in (a) to get the density function of the random variable $\mathrm{F}(Y_n) - \mathrm{F}(Y_1)$. (15%)

2. Let $X_1, X_2, \cdots, X_n$ be i.i.d. random variables with pdf

$$\mathrm{f}(x) = \frac{1}{6}\theta^4 e^{-\theta x} x^3; x \geq 0, \theta > 0.$$

   (a) Show that $\frac{3}{X_1}$ is an unbiased estimator of θ. (10%)

   (b) Find the joint distribution of X_1 and $\sum_{i=1}^n X_i$ and hence find the conditional density of X_1 given $\sum_{i=1}^n X_i$. (10%)

   (c) Show that the UMVUE of θ is $\mathrm{E}\left(\frac{3}{X_1} \mid \sum_{i=1}^n X_i\right)$ and compute the conditional expectation. (10%)

   (d) Find the Cramer-Rao lower bound for the variance of an unbiased estimator of θ. Does the variance of UMVUE attain the Cramer-Rao lower bound? (10%)

   (e) Show that $\frac{X_1}{\sum X_i}$ and $\sum X_i$ are independent random variables and hence show that $\mathrm{E}\left(\frac{X_1}{\sum X_i}\right) = \frac{1}{n}$. (10%)

3. Suppose that $X_1, X_2, \cdots, X_n$ be i.i.d. Poisson random variables with parameter λ_1. Independent variables $Y_1, Y_2, \cdots, Y_n$ are i.i.d. Poisson with parameter λ_2.

   (a) Show that the conditional distribution of $\sum X_i$, given that $\sum X_i + \sum Y_i = l$, is Binomial $\left(l, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$. (10%)

   (b) How could you use the binomial distribution in (a) to test $H_0: \lambda_1 = \lambda_2$ vs. $H: \lambda_1 \neq \lambda_2$? (15%)

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國立政治大學 109 學年度 碩士暨碩士在職專班 招生考試試題

考試科目	統計方法	系所別	統計學系	考試時間	2 月 7 日(五) 第四 節

1. (20%) A box contains 5 black balls and 10 white balls. Each of Player A and Player B is asked to draw 3 balls from the box and record the number of black balls obtained. Each of them does this 100 times, with the following results:

	Observed Outcomes			
	0	1	2	3
Player A	25	40	25	10
Player B	40	40	15	5

(a) (10%) Use $\alpha = 0.10$ to test the hypothesis that Player A drew the balls without replacement and Player B drew the balls with replacement.

(b) (10%) Use $\alpha = 0.10$ to test the hypothesis that the two multinomial populations are the same.

2. (20%) The following data were collected from a training program. 3 persons participated in this study and 3 courses were used to evaluate their resulting performances. The grades are shown below.

	Person 1	Person 2	Person 3
Course A	84	83	85
Course B	79	77	81
Course C	80	74	77

(a) (8%) At 95% confidence using the p-value approach, do persons perform differently?

(b) (8%) Using the critical value approach and $\alpha = 0.05$, are there any significant differences among the 3 courses?

(c) (4%) A researcher said "one-way ANOVA cannot be applied to the problem in (b)." Do you agree with this comment? Please give your reason.

3. (25%) Let $(y_1, x_1), (y_2, x_2), \ldots, (y_n, x_n)$, and $(w_1, x_1), (w_2, x_2), \ldots, (w_n, x_n)$, be two sets of independent observations where x_i, $i = 1, \ldots, n$, are fixed constants. Suppose we fit the model $y_i = \alpha_1 + \beta x_i + \epsilon_i$ to the first data set and the model $w_i = \alpha_2 + \beta x_i + \eta_i$ to the second data set. Suppose that ϵ_i and η_i are independently distributed with mean zero and variance σ^2.

(a) (15%) Find the least squares estimates of α_1, α_2 and β.

(b) (10%) Following Part (a), how to use t test to test $H_0: \beta = 0$ vs. $H_1: \beta \neq 0$? Please state your testing procedure(s) clearly to get full credits.

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4. (25%) Bowl A contains 100 red balls and 200 white balls; Bowl B contains 200 red balls and 100 white balls. Let p denote the probability of drawing a red ball from a bowl, but say p is unknown, since it is unknown whether Bowl A or Bowl B is being used. We shall test the null hypothesis H_0: $p = 1/3$ against H_1: $p = 2/3$.

(a) (4%) First, we draw three balls at random, one at a time and with replacement, from the selected bowl. Let X be the number of red balls drawn. Then we decide to reject H_0 if $X \geq 2$. Please give a reason to explain the rejection area should be $\{X \geq 2\}$ rather than $\{X \leq 2\}$.

(b) (6%) Please find the probabilities of making type I and type II errors in (a).

(c) (15%) As you can see in (b), the errors are too large, so we decide to draw more balls, say n balls, and the rejection area is $\{X \geq nr\}$, where X is the number of red balls from the n draws and r is the proportion and between 0 and 1. Please find the values of n and r, so that the type I error probability is 5% and the power is 90%. (Ps. Use the central limit theorem to find the approximate type I and II errors.)

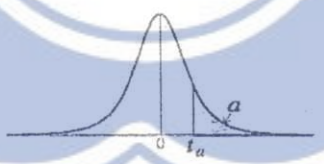
5. (10%) Let $X_1, X_2, \ldots, X_9$ be a random sample from the normal distribution $N(6,25)$, and let $\bar{X}$ and S^2 denote the sample mean and sample variance respectively. Find each of the following:

(a) (5%) $P(1.860 < \frac{3(\bar{X}-6)}{S})$

(b) (5%) $P(S^2 \leq 34.4688)$

Critical Values of the t Distribution

A table entry is the value of t_a, having an area to the right of a under a t distribution with *df* degrees of freedom.



df	$t_{0.20}$	$t_{0.15}$	$t_{0.10}$	$t_{0.05}$	$t_{0.025}$	$t_{0.01}$	$t_{0.005}$	$t_{0.001}$	$t_{.0005}$
1	1.376	1.963	3.078	6.314	12.71	31.82	63.66	318.3	636.6
2	1.061	1.386	1.886	2.920	4.303	6.965	9.925	22.33	31.60
3	0.978	1.250	1.638	2.353	3.182	4.541	5.841	10.21	12.92
4	0.941	1.190	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	0.920	1.156	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	0.906	1.134	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	0.896	1.119	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	0.889	1.108	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	0.883	1.100	1.383	1.833	2.262	2.821	3.250	4.297	4.781

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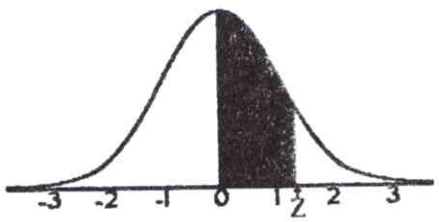

STANDARD NORMAL TABLE (Z)

Entries in the table give the area under the curve between the mean and *z* standard deviations above the mean. For example, for $z = 1.25$ the area under the curve between the mean (0) and z is 0.3944.

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0190	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2969	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3513	0.3554	0.3577	0.3529	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990
3.1	0.4990	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4992	0.4993	0.4993
3.2	0.4993	0.4993	0.4994	0.4994	0.4994	0.4994	0.4994	0.4995	0.4995	0.4995
3.3	0.4995	0.4995	0.4995	0.4996	0.4996	0.4996	0.4996	0.4996	0.4996	0.4997
3.4	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4998

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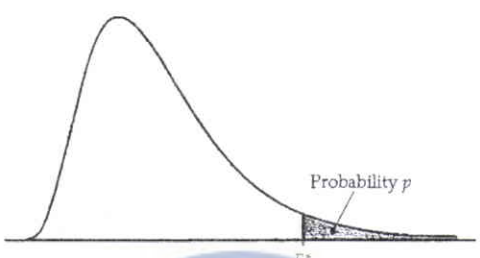


Table entry for p is the critical value F^* with probability p lying to its right.

TABLE E

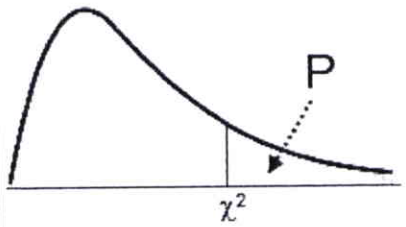
F critical values

Degrees of freedom in the denominator	p	Degrees of freedom in the numerator: 1	2	3	4	5	6	7	8	9
1	.100	39.86	49.50	53.59	55.83	57.24	58.20	58.91	59.44	59.86
	.050	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54
	.025	647.79	799.50	864.16	899.58	921.85	937.11	948.22	956.66	963.28
	.010	4052.2	4999.5	5403.4	5624.6	5763.6	5859.0	5928.4	5981.1	6022.5
	.001	405284	500000	540379	562500	576405	585937	592873	598144	602284
2	.100	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.38
	.050	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38
	.025	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.39
	.010	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39
	.001	998.50	999.00	999.17	999.25	999.30	999.33	999.36	999.37	999.39
3	.100	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.24
	.050	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81
	.025	17.44	16.04	15.44	15.10	14.88	14.73	14.62	14.54	14.47
	.010	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35
	.001	167.03	148.50	141.11	137.10	134.58	132.85	131.58	130.62	129.86
4	.100	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.95	3.94
	.050	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00
	.025	12.22	10.65	9.98	9.60	9.36	9.20	9.07	8.98	8.90
	.010	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66
	.001	74.14	61.25	56.18	53.44	51.71	50.53	49.66	49.00	48.47
5	.100	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.32
	.050	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77
	.025	10.01	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68
	.010	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16
	.001	47.18	37.12	33.20	31.09	29.75	28.83	28.16	27.65	27.24
6	.100	3.78	3.46	3.29	3.18	3.11	3.05	3.01	2.98	2.96
	.050	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10
	.025	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52
	.010	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98
	.001	35.51	27.00	23.70	21.92	20.80	20.03	19.46	19.03	18.69
7	.100	3.59	3.26	3.07	2.96	2.88	2.83	2.78	2.75	2.72
	.050	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68
	.025	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82
	.010	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72
	.001	29.25	21.69	18.77	17.20	16.21	15.52	15.02	14.63	14.33

備註	一、作答於試題上者，不予計分。 二、試題請隨卷繳交。

國立政治大學 109 學年度 碩士暨碩士在職專班 招生考試試題

考試科目	統計方法	系所別	統計學系	考試時間	2 月 7 日(五) 第四 節



Upper tail		0.3	0.2	0.1	0.05	0.02	0.01	0.005	0.001
df	2	2.41	3.22	4.61	5.99	7.82	9.21	10.60	13.82
	3	3.66	4.64	6.25	7.81	9.84	11.34	12.84	16.27
	4	4.88	5.99	7.78	9.49	11.67	13.28	14.86	18.47
	5	6.06	7.29	9.24	11.07	13.39	15.09	16.75	20.52
	6	7.23	8.56	10.64	12.59	15.03	16.81	18.55	22.46
	7	8.38	9.80	12.02	14.07	16.62	18.48	20.28	24.32
	8	9.52	11.03	13.36	15.51	18.17	20.09	21.95	26.12
	9	10.66	12.24	14.68	16.92	19.68	21.67	23.59	27.88
	10	11.78	13.44	15.99	18.31	21.16	23.21	25.19	29.59
	11	12.90	14.63	17.28	19.68	22.62	24.72	26.76	31.26
	12	14.01	15.81	18.55	21.03	24.05	26.22	28.30	32.91
	13	15.12	16.98	19.81	22.36	25.47	27.69	29.82	34.53
	14	16.22	18.15	21.06	23.68	26.87	29.14	31.32	36.12
	15	17.32	19.31	22.31	25.00	28.26	30.58	32.80	37.70
	16	18.42	20.47	23.54	26.30	29.63	32.00	34.27	39.25
	17	19.51	21.61	24.77	27.59	31.00	33.41	35.72	40.79
	18	20.60	22.76	25.99	28.87	32.35	34.81	37.16	42.31
	19	21.69	23.90	27.20	30.14	33.69	36.19	38.58	43.82
	20	22.77	25.04	28.41	31.41	35.02	37.57	40.00	45.31
	25	28.17	30.68	34.38	37.65	41.57	44.31	46.93	52.62
	30	33.53	36.25	40.26	43.77	47.96	50.89	53.67	59.70
	40	44.16	47.27	51.81	55.76	60.44	63.69	66.77	73.40
	50	54.72	58.16	63.17	67.50	72.61	76.15	79.49	86.66

備註	一、作答於試題上者，不予計分。 二、試題請隨卷繳交。